3.5 CLAIRAUT'S EQUATION

An equation of the form $y = px + f(p)$ ...(1)

is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$, we have $\dfrac{dp}{dx} = 0$

Integrating, $p = c$

Putting $p = c$ in (1), the required solution is $p = cx + f(c)$

Thus, the solution of Clairaut's equation is obtained by writing c for p.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $(y - px)(p - 1) = p$.

Sol. The given equation can be written as

$$y - px = \frac{p}{p-1} \quad \text{or} \quad y = px + \frac{p}{p-1}$$

This is of Clairaut's form. Hence putting c for p, the solution is $y = cx + \dfrac{c}{c-1}$.

Note. Many differential equations can be reduced to Clairaut's form by suitably changing the variables.

Example 2. *Solve* $e^{4x}(p-1)+e^{2y}p^2=0$.

Sol. [In problems involving e^{lx} and e^{my}, put $X=e^{kx}$ and $Y=e^{ky}$, where k is the H.C.F. of l and m].

Put $X=e^{2x}$ and $Y=e^{2y}$

so that $dX=2e^{2x}\,dx$ and $dY=2e^{2y}\,dy$

$$\therefore \quad p=\frac{dy}{dx}=\frac{e^{2x}}{e^{2y}}\frac{dY}{dX}=\frac{X}{Y}P, \quad \text{where } P=\frac{dY}{dX}$$

The given equation becomes $X^2\left(\frac{X}{Y}P-1\right)+Y\cdot\frac{X^2}{Y^2}P^2=0$.

or $\quad XP-Y+P^2=0 \quad$ or $\quad Y=PX+P^2$ which is of Clairaut's form.

$\therefore$ Its solution is $Y=cX+c^2$ and hence $e^{2y}=ce^{2x}+c^2$.

Example 3. *Solve* $(px-y)(py+x)=2p$.

Sol. Put $X=x^2$ and $Y=y^2$ so that $dX=2x\,dx$ and $dY=2y\,dy$

$$\therefore \quad p=\frac{dy}{dx}=\frac{x}{y}\cdot\frac{dY}{dX}=\frac{\sqrt{X}}{\sqrt{Y}}P, \quad \text{where } P=\frac{dY}{dX}$$

The given equation becomes

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}P\cdot\sqrt{X}-\sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\cdot\sqrt{Y}+\sqrt{X}\right)=2\frac{\sqrt{X}}{\sqrt{Y}}P$$

or
$$(PX-Y)(P+1)=2P \quad \text{or} \quad PX-Y=\frac{2P}{P+1}$$

or
$$Y=PX-\frac{2P}{P+1} \text{ which is of Clairaut's form.}$$

$\therefore$ Its solution is $Y=cX-\dfrac{2c}{c+1}$ and hence $y^2=cx^2-\dfrac{2c}{c+1}$.

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y=xp+\dfrac{a}{p}$
2. $y=px+\sqrt{a^2p^2+b^2}$
3. $\sin px\cos y=\cos px\sin y+p$
4. $xp^2-yp+a=0$
5. $(x-a)p^2+(x-y)p-y=0$
6. $p=\log(px-y)$
7. $p=\sin(y-px)$
8. $p^2(x^2-1)-2pxy+y^2-1=0$
9. $e^{3x}(p-1)+p^3e^{2y}=0$
10. $x^2(y-px)=yp^2$
11. $(y+px)^2=x^2p$.

Answers

1. $y=cx+\dfrac{a}{c}$
2. $y=cx+\sqrt{a^2c^2+b^2}$
3. $y=cx-\sin^{-1}c$
4. $y=cx+\dfrac{a}{c}$
5. $y=cx-\dfrac{ac^2}{c+1}$
6. $y=cx-e^c$
7. $y=cx+\sin^{-1}c$
8. $(y-cx)^2=1+c^2$
9. $e^y=ce^x+c^2$
10. $y^2=cx^2+c^2$ [**Hint.** Put $x^2=X$, $y^2=Y$]
11. $xy=cx-c^2$. [**Hint.** Put $xy=v$]

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Solve the following differential equations (Clairaut's Equation)

6. $y = xp + \frac{a}{p}$ **[M.D.U. 2003; K.U. 2013]** **7.** $(y-px)(p-1) = p$

8. $(y-px)^2 = 1+p^2$ **9.** $p = \tan(px-y)$ **[K.U. 2016, 15, M.**

10. $y = px + \sqrt{a^2p^2+b^2}$ **[M.D.U. 2015]** **11.** $p = \log(px-y)$

12. $y = x\frac{dy}{dx} + e^{\frac{dy}{dx}}$ **13.** $(y-px)(p+1) = p^2$ **[M.**

Answers

1. $y = 2 - p^2 + ce^{-p}(1+p)$ and together with given relation.

2. $y = -ap + \frac{1}{\sqrt{1-p^2}}(a\sin^{-1}p + c)$ and together with given relation.

3. $y = c(x-b) + \frac{a}{c}$ **4.** $y = (x+c)\tan\frac{x}{2} + 1$

5. $ax = \frac{3}{2}p^2 - mp + m^2\log(p+m) + c$, $ay = \left[-m\left(\frac{3}{2}p^2 - mp + m^2\log(p+m) + c\right) + mp\right.$

6. $y = cx + \frac{a}{c}$ **7.** $y = cx + \frac{c}{c-1}$ **8.** $y = cx \pm \sqrt{1+c^2}$

9. $y = cx - \tan^{-1}c$ **10.** $y = cx + \sqrt{a^2c^2+b^2}$ **11.** $y = cx - e^c$

12. $y = cx + e^c$ **13.** $y = cx + \frac{c^2}{c+1}$

4.2 EQUATIONS REDUCIBLE TO CLAIRAUT'S FORM :

Some differential equations of first order and higher degree can be reduc Clairaut's form by making suitable substitutions.

There is no general method to find the proper substitutions. These can be learn practice only.

Illustrative Examples

Example 1 : Solve $e^{5x}(p-1) + p^5e^{4y} = 0$.

Solution : The given differential equation is

$$e^{5x}(p-1) + p^5e^{4y} = 0$$

Put $X = e^x$ and $Y = e^y$

$\Rightarrow$ $dX = e^x dx$ and $dY = e^y dy$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{e^y dy}{e^x dx} = \frac{e^y}{e^x}\frac{dy}{dx} = \frac{e^y}{e^x}.p$$

Assume $$P = \frac{dY}{dX} = \frac{e^y}{e^x} \cdot p = \frac{Y}{X} \cdot p$$

$\Rightarrow$ $$p = \frac{X}{Y} P$$

Thus from equation (1), we have

$$X^5\left[\left(\frac{X}{Y} P\right) - 1\right] + \frac{X^5}{Y^5} P^5 Y^4 = 0$$

or $$\frac{X}{Y} P - 1 + \frac{P^5}{Y} = 0$$

or $$PX - Y + P^5 = 0$$

or $$Y = PX + P^5$$

which is a clairaut's equation. Therefore the solution is given by

$$Y = cX + c^5 \qquad \text{[By replacing } P \text{ by } c]$$

or $$e^y = ce^x + c^5$$

Example 2 : Solve $x^2 (y - px) = yp^2$

Solution : The given differential equation is

$$x^2(y - px) = yp^2 \qquad \text{.......(1)}$$

Let us assume, $x^2 = X$ and $y^2 = Y$

$\therefore$ $2xdx = dX$ and $2y\,dy = dY$

$\therefore$ $$p = \frac{dy}{dx} = \frac{x\,dY}{y\,dX} = \frac{x}{y} P, \quad \text{where } P = \frac{dY}{dX}$$

Putting the value of p in equation (1), we get

$$x^2\left(y - \frac{x}{y} P.x\right) = y\frac{x^2}{y^2} P^2$$

or $$x^2(y^2 - x^2 P) = x^2 P^2$$

or $$X(Y - XP) = XP^2 \qquad [\because x^2 = X \text{ and } y^2 = Y]$$

or $$Y = PX + P^2$$

which is Clairut's equation. So its solution is given by

$$Y = cX + c^2 \qquad \text{[P is replaced by c]}$$

or $$y^2 = cx^2 + c^2$$

Exercise – 2.5

Solve the following differential equations by reducing them to Clairaut's form

1. $(px - y)(x + py) = a^2 p$
2. $(y + px)^2 = x^2 p$
3. $(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + yp) + (x + yp)^2 = 0$
4. Reduce the equation $y^2 (y - xp) = x^4 p^2$ to Clairaut's equation by using the substitution $x = \frac{1}{X}, y = \frac{1}{Y}$ and hence solve the equation.

5. Solve $e^{4x}(p-1)+e^{2y}p^2=0$ by using the substitution $X=e^{2x}$, $Y=e^{2y}$

6. Solve $(px^2+y^2)(px+y)=(p+1)^2$ by using the substitution $X=x+y$, $Y=xy$

[M.D.U. 2013, 14]

7. Solve the differential equation $e^{3x}(p-1)+p^3e^{2y}=0$

Answers

1. $y^2=cx^2-\dfrac{a^2c}{c+1}$

2. $xy=cx-c^2$ [Hint : Put $x=X, xy=Y$]

3. $x^2+y^2=c(x+y)-\dfrac{c^2}{4}$ [Hint : Put $x+y=X$, $x^2+y^2=Y$]

4. $x=cy+c^2xy$ **5.** $e^{2y}=ce^{2x}+c^2$ **6.** $xy=c(x+y)-\dfrac{1}{c}$

7. $e^y=ce^x+c^3$

Def. Singular Solution : A singular solution is a solution of the differential equation